

# Issues in Contemporary Metaphysics

## Lecture 3: Numbers

### 1. Realism About Numbers

We might be Platonists about numbers (say they're eternal, outside space and time and what we refer to when we make mathematical statements). So '2+3=5' is true, according to the Platonist, because in Platonic Heaven there exists an abstractum, the number 2, that stands in the '\_\_\_ plus \_\_\_ equals \_\_\_' relation to the number 3 and the number 5. (And if you went in for that sort of thing, we'd say that this state of affairs was the *truthmaker* for that sentence/proposition.)

There are other forms of realism, though. *Constructivists* believe we create the numbers – they exist but are only the product of our minds. *Intuitionists* are the most commonly discussed version of these. Intuitionism is a bit different from a more vanilla Platonism. Take a mathematical proposition like Goldbach's Conjecture:

Every number greater than 2 is the sum of two prime numbers.

We don't know whether that's true or false. For every number checked so far, it's true (where we've checked numbers up to 4,000,000,000,000,000,000 – that's 4 quintillion) But we don't know whether it's true for *every* number. The Platonist thinks the matter is settled though. Whether we *know* whether it's true or false it *is* true or false. Out there, in Platonic heaven, all of the numbers stand in certain relations – and if any one of them fails to be the sum of two primes, the Conjecture is false and if not then it's true. But the intuitionist doesn't *have* those things – if you haven't thought of that number it doesn't exist, and since it doesn't it the facts about what properties it has (e.g. being the sum of two primes) isn't fixed.

So they think that the truth of Goldbach's Conjecture isn't settled. This ends up being a counterexample to the Law of Excluded Middle ( $P \vee \sim P$ ). For they don't think it is the case, but it isn't the case either. They accept *intuitionistic logic* wherein the law of excluded middle fails (and  $\sim\sim P$  doesn't imply  $P$ ).

### 2. Set Theory

In English when we talk about sets of things we might be talking about *collections* of things. In metaphysics when we talk about a 'set' it has a technical meaning. The sets that we are talking about are the same ones that *set theory* (a branch of mathematics) deals with. The set of, say, you and I is something *in addition* to you and I. So a set is a thing, usually taken to be an abstract object (like those pesky universals!). We represent this set by

{ you, me }

You can also have sets with just one thing:

{ you }                      or                      { me }

Sets of sets:

{ { you } , { me } }

And sets with no members!

Represented as { } or  $\emptyset$

So in set theory we get sets with nothing in them (the null set), and can have sets of sets. But you don't have empty collections, or collections of collections. Sets aren't collections! So most philosophers treat sets as abstract objects – new things, labelled with a familiar English word but not, themselves, familiar things.

Set theory is the basis of mathematics. Indeed, some mathematics can only be done in set theory (for instance certain proofs concerning Diophantine Equations) And some have thought that we should *identify* numbers with sets.

#### The Zermelo reduction

Take that 'empty set'. Identify that with the number 0

$0 = \emptyset$

Next, say that the successor of any number  $n$  is its *singleton*. The singleton of something is simply the set which has only it as a member. So my singleton is: { Nikk } That set has a singleton: { { Nikk } }

So  $n = \{ n-1 \}$

$1 = \{ \emptyset \}$

$2 = \{ \{ \emptyset \} \}$

$3 = \{ \{ \{ \emptyset \} \} \}$

#### The von Neumann reduction

Take that 'empty set'. Identify that with the number 0

$0 = \emptyset$

Next, say that the successor of any number  $n$  is the set of all of its predecessors. So 1 is preceded by 0 so is the set of just 0

{  $\emptyset$  }

2 is preceded by 0 and 1 so is the set of both of them

{  $\emptyset$ , {  $\emptyset$  } }

3 is preceded by 0, 1 and 2 so is the set of all three

{  $\emptyset$ , {  $\emptyset$  }, {  $\emptyset$ , {  $\emptyset$  } } }

We're reducing numbers to sets. Do we get any benefits out of it? When we reduced properties to objects or concepts it was more parsimonious! But maybe not here – if you were a nominalist you're not going to believe in sets either. So

when someone says ‘I’ve got a parsimonious theory that reduces numbers to sets’ the nominalist will cry: ‘I don’t like sets *either*’ One way to mitigate this cost would be to demonstrate that sets are sensible objects that are already in our ontology. So Black and Maddy try and show how sets are, in fact, quite sensible entities such as collections – contrary to what I’ve just said. Another way is to show that introducing sets is useful in *other* areas of ontology.

#### *Properties Without Universals (again)*

Rather than believing in universals, we could take properties to be sets. Which sets? The property ‘blue’ can’t just be some random set (say of you and me). It’s the set of all blue things! ‘Honesty is a virtue’ does commit us to the existence of some property ‘Honesty’. But it’s the set of all honest people! So we admit that properties exist, but we identify them with (allegedly) more respectable entities than universals. But it has problems.

Sets, we said, are identical iff they have the same members. Now take the properties ‘cordate’ and ‘renate’. All cordates are renates. So if ‘cordate’ is the set of all cordates, and ‘renate’ is the set of all renates, then those sets have the same members. Same members means they’re identical. But surely this is false! They *aren’t* the same property!

Another example. The honest people in the world and the terrorists aren’t one and the same. Pretend that the terrorists are all honest. But now imagine that the terrorists win, and kill everyone else. Now the terrorists and the honest people are the same people. But that means that whilst in our world *being honest* and *being a terrorist* are different properties, in *that* world they are one and the same! So if ‘Honesty is a virtue’ is true then, as *being honest* is identical to *being a terrorist* then ‘Being a terrorist is a virtue’ is true! That’s not right!

### **3. Arguments for Realism**

#### *Abstract Reference*

As with properties – and everything else – there seem to be commitments to numbers in our everyday language. Moreover, it’s not just solely mathematical statements that are worrisome.

For instance:

There are more cats than dogs

James is Nikk’s ancestor

Generally it’s thought that sentences like these have truth conditions involving sets. Why? Well, you can say that there’s two cats and one dog.

$\exists x \exists y_1 \exists y_2 (Dx_1 \ \& \ Cy_1 \ \& \ Cy_2 \ \& \ x_1 \neq y_1 \ \& \ x_1 \neq y_2 \ \& \ y_1 \neq y_2)$

You can say there’s three cats and two dogs.

$\exists x \exists x_2 \exists y_1 \exists y_2 \exists y_3 (Dx_1 \ \& \ Dx_2 \ \& \ Cy_1 \ \& \ Cy_2 \ \& \ Cy_3 \ \& \ x_1 \neq y_1 \ \& \ x_1 \neq y_2 \ \& \ y_1 \neq y_2 \ \& \dots)$

But how can you say, *in general*, that there are more cats than dogs? Frege conscripted in sets for the task: it’s just to say that the set of cats has a greater number of members than the set of dogs. ‘More than’ talk is analysed using sets. But perhaps it can be paraphrased away – we can say there are more cats than dogs without ever mentioning sets or numbers. Quine and Goodman tried just such a thing. Say that something ‘is a bit’ if and only if it’s as big as the smallest cat or dog. So every cat or dog will have a bit as a part. Then say that there are more cats than dogs iff the object composed out of all the bits that are parts of the cats is bigger than the objects composed out of all the bits that are parts of the dogs.

There are more cats than dogs =df Every object that has (as a part) one bit from each cat is bigger than some object that has (as a part) one bit from each dog.

#### *Quine-Putnam Indispensability Argument*

A more pressing argument comes from Quine and Putnam. They say that numbers are indispensable to science. For instance, scientists need to say things like:

The surface area of Saturn is 1.08 x 10<sup>12</sup> km<sup>2</sup>.

And that, in first order logic is:

1.08 x 10<sup>12</sup> is-the-surface-area-in-km<sup>2</sup>-of Saturn

$\exists x \exists y (x = 1.08 \times 10^{12} \ \& \ y = \text{Saturn} \ \& \ xRy)$

Where R is ‘\_\_ is-the-surface-area-in-km<sup>2</sup>-of \_\_’

So it seems there is a commitment to numbers in scientific theories. And sentences like those are indispensable for science – for how else can we predict things about Saturn without statements about its surface area? If it’s indispensable to science then it must exist! If we know anything, so the argument goes, we know we should trust our best scientific theories. So if the best scientific theory demands X then X exists. So if it demands numbers, numbers exist!

Other examples. We might explain why we can't tile a rectangular region with exactly 191 tiles *because* 191 is a prime number. You get similar examples in biology. Life cycles in prime numbers can be common. Certain flies mate, and then their children come from the soil a prime number of years later. Why? Well, to avoid being eaten. If your predator went a-hunting every 6 years and your life cycle was 12 years then you'd get predated every time the flies gestated. If your life cycle was 15 years you'd get predated every two cycles (miss the first, get the second) At 13 years you will get predated every *six* life cycles. But now it seems that prime numbers play a role in biology. The insects have a thirteen year life cycle because 13 is a prime number.

Some responses attack the idea that just because something is indispensable to science means it exists. Others, such as Harty Field, attack the idea that numbers are indispensable to science. Sure they're *useful*, but indispensable? Like the economist and the average man, we might be able to nominalise them away and offer paraphrases for them.

#### 4. Benacerraf's Problems

There are the usual suspects for not believing in numbers. If they existed they'd be abstract, and who wants to believe in abstracta? Indeed, worries like the *epistemological concern* (back from lecture 1) – that if abstracta existed how would we *know* they existed – get their clearest exposition in the works of Benacerraf. In his work he posed a problem for realists about numbers – if they existed, how would we know about them?

Time now to consider some responses. We might, for instance, make numbers located. Just as Armstrong locates Universals in spacetime ('immanent universals') we might say that sets or numbers are, in fact, in space and time and we have knowledge of them that way. Maddy has such a theory. For Maddy, mathematics is based on set theory. And sets, for Maddy recall, are just collections. So { Nikk , Projector } is just located where I am and the projector is. And we know about collections! Our brains are well equipped to understand collections. So we have immediate epistemic access to the entities – the sets – that are the basis of all mathematics.

Gödel takes a different tactic. Set theory has various axioms. That whenever you have some things, you have a set of those things; if two sets have the same members they're the same set etc. Gödel just thought these things were obviously true. So even though they are far off, a crude reading would seem to suggest that Gödel thinks we have some sort of sixth sense for number detection. You might want to look into more sophisticated readings of what Gödel is trying to say!

#### 5. Formalism

We've, briefly, looked at some nominalist alternatives. Quine and Goodman removed commitment by offering paraphrases without quantifying over numbers. Field, as I say, does the same. There are different nominalisms, however. Let's consider nominalist strategies in general. We might remove an unwanted commitment to the *xs* by showing how the sentence that appears to quantify over the *xs* does not in fact do so. We might remove the unwanted commitment by showing that there is a commitment, but it isn't that bad.

One further way to do this would be to say that all talk about numbers is, in fact, about *numerals*. So when I say

The square of 2 = 4

I *don't* mean that some abstract object (that which is 22) is identical to some other abstract object (4). No: I mean that the *numerals* on the page are identical. This is bonkers though – the bit on the left *isn't the same as the bit on the right*.

But wait, maybe there's something close to this that might help. One such theory is *formalism*. Formalists think that it's all about the numerals, the ink blots on the page, the pixels on the computer, the projected light on the screen etc. Call such things 'concatenations' (the term is from Quine and Goodman)

The formalist says that what's important to a mathematician is that '2+2=4' follows from the axioms of arithmetic. What's important is that, in set theory

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$$

counts as an axiom (and all of the other axioms count as axioms, and all of the theorems count as theorems etc.)

But you don't need sets or numbers for *that* to happen. For instance, take Sudoku. If a line of Sudoku read:

1 3 4 2 6 7 5 8 9

Then that's an acceptable Sudoku line. If a line read

1 1 2 2 3 3 4 4 5

then it's unacceptable. That string of numerals – *that concatenation* – does not fall under the predicate '\_\_\_ is a legitimate string of numbers in Sudoku'. And we could do that for all combinations of numbers, perhaps laying down the rules for that counts as an acceptable Sudoku grid.

Note that it doesn't really make any sense to say that the numbers are true or false either. The concatenation '1 3 4 2 6 7 5 8 9' is an acceptable combination of numerals in Sudoku but it *isn't* true or false. It doesn't assert anything to be true

or false! We can do the same for statements in maths and set theory. We could say that those statements are correct mathematical statements, without thinking that they are true, or commit us to the existence of numbers or anything like that. Remember, only statements that are true carry a corresponding ontological commitment.

Similarly, we could say that every time a mathematician says ‘This is a theorem of set theory’ and points at some symbols on a page or ‘This is a mathematical proof’ and points at a collection of lines that make up a proof, they’ve said something true. That concatenation – *those ink blots on the page* – are an axiom. That concatenation – *those ink blots on the page that make up a book length proof* – are a proof. But the concatenations themselves aren’t true or false. They’re axioms, they’re theorems, but they’re not asserting anything. So maths is just like a game on this view. In the same way that a chess move might be acceptable in a game, and a line of numbers acceptable to be entered into a Sudoku, we don’t mistakenly think we’re asserting truth-apt sentences.

This is, of course, a form of nominalism. Now we remove commitment to numbers by saying that it’s as odd to think mathematicians are talking ‘about numbers’ as it is to think Sudoku players are engaging in meaningful debates about whether their grids are right or not. They might be right or not *within the confines of the game* but that’s as good as it gets. One has to say something more than this. One needs to define – without mentioning numbers! – what it is for something to be an axiom, or a theorem; the conditions under which one concatenation follows from another; the conditions under which a concatenation counts as a proof etc. And that has been attempted. See Quine and Goodman’s 1947 paper for an example.